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Traits: The Formal Model

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Abstract. Traits are reusable units of behaviour that provide a level of structuring for object-oriented programs above the level of methods but below the level of classes. Structuring classes with single-inheritance alone can lead to duplicated code when different branches of the hierarchy need to use the same feature. Multiple-inheritance and mixins alleviate this problem, but lead to other difficulties in the face of evolution: changes to classes or mixins can break code lower in the hierarchy in unexpected ways.

Traits solve both problems by factoring out shared behaviour as sets of methods that do not depend on state. Trait composition is symmetric and conflicts must be managed explicitly; this means that changes in the components do not lead to unexpected side effects.

This paper presents a formal model of traits, and defines some basic properties of traits and classes. We also model the internal dependencies created by self and super-sends so that we can specify precisely when two classes are equivalent.

1 Introduction

Traits provide a mechanism that can be used to factor out shared behaviour from classes. The purpose of this paper is to provide a formal basis for understanding and reasoning about traits; it is not intended to be self-contained. A companion paper, which we will refer to as the “traits paper” [SDNB02], shows that traits overcome many difficulties with single and multiple inheritance, and with mixins. We have also reported on the use of an experimental implementation to refactor a part of the Smalltalk collections hierarchy [BSD02].

The semantics of single inheritance is well understood and it is described by several operational and denotational formalizations [GR83,CP89]. For conciseness, we do not repeat any details of these formalizations. Instead we focus on two contributions. First, we define traits and the various operators that can be used to combine traits into compound traits, and ultimately into classes. Second, we make precise the internal dependencies within a class that are created when methods send messages to self and super. This enables us to specify when

different ways of composing a class are equivalent from the perspective of provided and required services. Depending on the programmer’s goals, equivalent classes might be regarded as legal refactorings, or as alternative views on a more abstract form of the program.

2 The Traits Language

While developing traits we have not focussed on issues of concrete syntax. In fact, because our initial implementation is part of the Smalltalk interactive development environment, we have been able to defer syntactic considerations almost entirely: traits are created and manipulated by the traits browser using method templates and menu selections. However, we have found that readers grasp the essentials of traits more quickly if they can see the various forms displayed as a grammar, so in the traits paper we presented an abstract syntax for the principal operations on traits [SDNB02, section 4]. Here we reproduce that grammar, and expand it to include constructs for method dictionaries, for the inheritance operation on classes, and for definitions that give names to classes and traits.

method dictionary:  \[ D ::= \{ \} | \{ x_1 \mapsto m_1, \ldots, x_n \mapsto m_n \} \]

trait expression:  \[ T ::= N_T | T + T | T - x | T[x \rightarrow y] \]

trait definition:  \[ T_{Def} ::= N_T = D | N_T = D \text{ with } T \]

fields:  \[ F ::= \{ \} | \{ f_1, f_2, \ldots, f_n \} \]

class expression:  \[ C ::= \text{nil} | N_C | F \text{ in } D \text{ extends } N_C \]
\[ | F \text{ in } D \text{ with } T \text{ extends } N_C \]

class definition:  \[ C_{Def} ::= N_C = C \]

\( D \) denotes a method dictionary, which is either empty or contains a finite number of services (i.e., bindings of labels to methods) in which all of the labels \( x_i \) are distinct. \( T \) denotes a trait expression, which can be a named trait \( N_T \), the symmetric composition \( T + T \) of two traits, or a trait that is subject to an exclusion \( T - x \) or an alias \( T[x \rightarrow y] \). A trait definition binds a trait name \( N_T \) to one of two forms: \( D \), a method dictionary, or \( N_T = D \text{ with } T \), a method dictionary that adds methods to (and possibly overrides methods from) a trait.

\( F \) is a possibly empty set of variable (or field) names; it is not important for our purposes whether these are instance variables, class variables, or static variables. \( C \) denotes a class, which can be nil, a named class \( N_C \), or a conventional class \( F \text{ in } D \text{ extends } N_C \), which consists of some fields \( F \), a method dictionary \( D \), and a named superclass \( N_C \). In addition, a class can be structured \( F \text{ in } D \text{ with } T \text{ extends } N_C \); the structured class acquires methods from the trait \( T \) augmented (and possibly overridden) by those from the explicit method dictionary \( D \). Finally, a class definition \( C_{Def} \) is simply a binding of a class name \( N_C \) to a class \( C \).

Since it is context free, this abstract syntax does not capture several context conditions. These are that a name \( N_T \) cannot be used as a trait expression until it has been defined (and similarly for \( N_C \)). Also, the form \( F \text{ in } D \) places the fields
defined in $F$ in scope within the method bodies $m_i$ that appear in $D$. Conversely, 
no fields are in scope in $D$ in the forms $N_T = D$ and $N_T = D$ with $T$.

The syntax does capture some important restrictions that the traits browser
imposes for reasons of human engineering. The with keyword, which allows
method overriding, is second class. It can appear only in the definition of a
named trait or class, and only with a literal method dictionary on the left hand
side; these restrictions help to ensure that all overriding of services in a trait is
explicit.

3 A Model for Traits

3.1 Informal Introduction

Because it is not important for the semantics of traits, we do not represent
the internal structure of methods in our model. Instead, we model methods as
atomic values in a set $\mathcal{M}$, augmented with top and bottom elements to create
a flat lattice; $\bot$ represents “no method” and $\top$ stands for a “method conflict”.
This also means that our model is independent of any particular programming
language.

Methods should be thought of as implementations, or code sequences. All we
assume about methods is that we can ascertain when they may send messages
to instances of their class. Such message sends are of two kinds: self-sends and
super-sends. Whereas self-sends employ the usual message lookup that starts in
the class of the current instance, super-sends cause the message lookup to start
in the superclass of the class that defines the method containing the super-send.

Traits are essentially sets of bindings of labels to methods; the labels name
the methods. We call these bindings services. Traits both provide services (i.e.,
the methods implemented) and require services (i.e., those that are invoked by
self- and super-sends). When we say that a class or a trait provides a service
with a particular label we mean that a binding of that label to a method exists.
When we say that a service with a particular label is required, we mean that a
binding of that label to a method needs to exist.

We introduce summation (+), exclusion (−), aliasing (→) and overriding (▷)
as mechanisms for composing and modifying traits. When traits are composed
with +, labels that are bound to different methods will conflict; we represent
this by binding the label to $\top$ in the sum. The $\triangleright$ operation is intended to be
used to override these bindings to $\top$ with proper methods.

Traits are subordinate to classes in several ways. Unlike classes, traits bind
neither self nor super. Classes can contain fields, and methods in classes can
refer to those fields, so it is possible for classes to provide accessor methods
(also known as setters and getters). In contrast, traits do not contain fields, and
as a consequence methods in traits cannot refer to fields. However, because the
requirements of a trait method can be met by accessor methods, a trait method
can access state indirectly.

A class is built up from traits, methods, and a single superclass using single
inheritance. If the traits conflict, the class can eliminate the conflict by overriding
the conflicting methods. super is bound only at the point that a class is composed, and self refers to an instance of a composed class.

The labels required by a class consist of the set of self-sends made by all the methods reachable by the super-send chain, minus the set of labels of the provided services. A class is concrete if all its requirements are fulfilled, otherwise it is abstract. A class is well-founded if the super-sends are properly bound; note that a well-founded class can still be abstract. Two classes are equivalent if they provide the same set of services, and all the methods reachable by sequences of self- and super-sends are the same.

In the remainder of this section we make this model formally precise. In section 4 we consider several examples from the traits paper, and show how they are represented in the model.

3.2 Methods

The primitive elements of our model are the following disjoint sets:

- \( \mathcal{M} \), a countable set of methods,
- \( \mathcal{L} \), a countable set of method labels, and
- \( \mathcal{F} \), a countable set of field labels.

Because of the context restriction mentioned in section 2, trait methods cannot access fields directly. In contrast, class methods can access fields, and the method dictionaries of classes can define accessor methods that can be used by trait methods to access fields indirectly. We do not make this distinction explicit in the model by partitioning \( \mathcal{M} \). Instead, it will be sufficient to model methods as atomic elements, and not to consider their internal structure. For instance, we make no assumption on how equality of methods is tested in an actual language: \( m_1 = m_2 \) might be equality of strings, of syntax trees, of bytecode, or of something else.

We are, however, interested in which other methods are required by a given method. We therefore model:

- \( \textit{selfSends} : \mathcal{M} \rightarrow 2^\mathcal{L} \), the set of labels used in self-sends, and
- \( \textit{superSends} : \mathcal{M} \rightarrow 2^\mathcal{L} \), the set of labels used in super-sends.

We extend \( \textit{selfSends} \) and \( \textit{superSends} \) to sets of methods by union:

- \( \textit{selfSends} : 2^\mathcal{M} \rightarrow 2^\mathcal{L} \)

\[
\textit{selfSends}(\mu) = \bigcup_{m \in \mu} \textit{selfSends}(m)
\]

- \( \textit{superSends} : 2^\mathcal{M} \rightarrow 2^\mathcal{L} \)

\[
\textit{superSends}(\mu) = \bigcup_{m \in \mu} \textit{superSends}(m)
\]
Note that the precise definitions of selfSends and superSends are language-dependent. For example, consider the method

\begin{verbatim}
fasten
    -- anObject -- self hook.
anObject := self.
anObject button.
self class new clip.
\end{verbatim}

It is clear that hook should be recognized as a self-send. What about button? A little local static analysis ought to be able to show that this is also a self-send. Is clip a self-send? We avoid answering this question! clip is a requirement on the class, but whether it is recognized as such depends on how thorough an analysis is carried out by the tools. We assume only that selfSends(m) provides a lower bound on the set of messages that an object might send to instances of its class while executing m. Other messages might be sent through indirect paths that are hard to discover.

We extend the set of methods $M$ to a flat lattice $M^\star$, with $\bot \sqsubseteq m \in M \sqsubseteq \top$, and in which all other elements are incomparable. We will use $\bot$ to represent undefined and $\top$ to represent a method conflict. Thus, the least upper bound or join operator $\sqcup$ for $M^\star$ is as shown:

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c}
  \bot & m_1 & m_2 & \top \\
\hline
  \bot & m_1 & m_2 & \top \\
  m_1 & m_1 & m_1 & \top & \top \\
  m_1 & m_2 & \top & m_2 & \top \\
  \top & \top & \top & \top & \top \\
\end{array}$$

\[\sqcup\]

**Definition 1.** A service is a partial function binding a single method label to a particular method. We use the notation:

$$x \mapsto m$$

for the service that binds the label $x \in \mathcal{L}$ to the method $m \in M$, and

$$x \mapsto \top$$

to indicate that label $x$ is in conflict.

### 3.3 Traits

**Definition 2.** A trait, $t \in T$, is a finite set of services, i.e., a function,

$$t : \mathcal{L} \rightarrow M^\star$$

where \{l \in \mathcal{L} | t(l) \neq \bot\} is a finite set.
Since traits are finite, we will always specify them extensionally, listing only bindings to elements $\mathcal{M}$ and $\top$. For example,

$$ t = \{ x \mapsto m_1, y \mapsto m_2, z \mapsto \top \} $$

defines a trait $t$ that maps label $x$ to method $m_1$, label $y$ to label $m_2$, and for which label $z$ has a conflict.

By convention, $\text{selfSends}$ and $\text{superSends}$ of $\top$ and $\bot$ are all $\emptyset$. We extend $\text{selfSends}$ and $\text{superSends}$ to traits in the obvious way:

- $\text{selfSends} : \mathcal{T} \to 2^\mathcal{L}$

  $$ \text{selfSends}(t) \overset{\text{def}}{=} \text{selfSends}(t(\mathcal{L})) $$

- $\text{superSends} : \mathcal{T} \to 2^\mathcal{L}$

  $$ \text{superSends}(t) \overset{\text{def}}{=} \text{superSends}(t(\mathcal{L})) $$

Since traits are just finite mappings, two traits are equal when these mappings are equal, that is, when equal labels map to equal methods.

**Definition 3.** The conflicts, $\text{conflicts} : \mathcal{T} \to 2^\mathcal{L}$, of a trait $t$ are defined by:

$$ \text{conflicts}(t) \overset{\text{def}}{=} \{ l \mid t(l) = \top \} $$

**Definition 4.** A method dictionary $d$ is a conflict-free trait, that is, a trait $d$ such that $\text{conflicts}(d) = \emptyset$.

Conflict-free traits model the literal method dictionaries $D$ of the traits language (see section 2). 

**Definition 5.** The provided labels, $\text{provided} : \mathcal{T} \to 2^\mathcal{L}$, of a trait $t$ are defined as:

$$ \text{provided}(t) \overset{\text{def}}{=} \{ l \in \mathcal{L} \mid t(l) \in \mathcal{M} \} $$

i.e., the set of all labels that $t$ does not map to $\bot$ or $\top$.

**Definition 6.** The required labels, $\text{required} : \mathcal{T} \to 2^\mathcal{L}$, of a trait $t$ are defined as:

$$ \text{required}(t) \overset{\text{def}}{=} \text{selfSends}(t) \setminus \text{provided}(t) $$

Notice that the required labels of a trait do not consider super-sends. However, when we compose classes from traits we must take super-sends into account in order to determine the required labels of the class (cf. Definition 17).
3.4 Trait Operations

Traits can either be specified directly, as lists of services, or they may be composed from other traits, by means of summation (+ : $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$), exclusion (− : $\mathcal{T} \times \mathcal{L} \to \mathcal{T}$), aliasing ([:=] : $\mathcal{T} \times \mathcal{L} \times \mathcal{L} \to \mathcal{T}$), and overriding ($\triangleright$ : $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$).

**Definition 7.** Summation composes two traits, forming the union of non-conflicting bindings and “disabling” the conflicting bindings:

$$(t_1 + t_2)(l) \overset{\text{def}}{=} t_1(l) \sqcup t_2(l)$$

For example:

$$\{a \mapsto m_1, b \mapsto m_2, c \mapsto m_3\} + \{a \mapsto m_1, b \mapsto m_3\} = \{a \mapsto m_1, b \mapsto \top, c \mapsto m_3\}$$

**Proposition 1.** Symmetric composition is associative and commutative.

**Proof.** Immediate from the definition, since the join operator $\sqcup$ is associative and commutative.

**Definition 8.** Exclusion removes a binding from a trait:

$$(t - l)(x) \overset{\text{def}}{=} \begin{cases} \bot & \text{if } x = l \\ t(x) & \text{otherwise} \end{cases}$$

For example:

$$\{a \mapsto m_1, b \mapsto \top\} - b = \{a \mapsto m_1\}$$

**Definition 9.** Aliasing introduces an additional name for an existing method:

$$t[x:=y](l) \overset{\text{def}}{=} \begin{cases} t(y) & \text{if } l = x \\ t(l) & \text{otherwise} \end{cases}$$

For example:

$$\{a \mapsto m_1, b \mapsto m_2\}[c:=b] = \{a \mapsto m_1, b \mapsto m_2, c \mapsto m_2\}$$

These three operations model the +, − and → operations from the language syntax of section 2.

When composing classes or traits, we may need to replace conflicts by proper methods. For this reason we introduce the $\triangleright$ operator.

**Definition 10.** A trait $t_1$ may replace some of the bindings in another trait $t_2$, as follows:

$$(t_1 \triangleright t_2)(l) \overset{\text{def}}{=} \begin{cases} t_2(l) & \text{if } t_1(l) = \bot \\ t_1(l) & \text{otherwise} \end{cases}$$

The $\triangleright$ operator models the with syntax of the traits language; the trait $d$ with $t$ is modeled by $d \triangleright t$. Because $\triangleright$ is asymmetric, in our concrete traits language its use is limited to definitions of named traits and named classes in which literal method definitions appear on the left hand side (see the discussion in Section 2).
3.5 Classes

Whereas traits are just sets of services without any context, a class binds self and super for its methods. In addition, a concrete class must provide all its required methods. Because the method referred to by a super-send depends on which methods are defined by the various superclasses, to define the meaning of “required” for a class we must model the entire inheritance chain of the class. We do this by representing a class as a sequence of method dictionaries.

**Definition 11.** A class, \( c \in \mathcal{C} \), is either the empty class, nil, or a sequence \( \langle \phi, t \rangle \cdot c' \), where \( \phi \subset \mathcal{F} \), \( t \in \mathcal{T} \), and \( c' \in \mathcal{C} \).

If we assume that \( s_m \) is the model for \( s \), then the conventional class

\[
\phi \text{ in } d \text{ extends } s
\]

is modeled by \( \langle \phi, d \rangle \cdot s_m \), and the structured class

\[
\phi \text{ in } d \text{ with } t \text{ extends } s
\]

is modeled by \( \langle \phi, d \triangleright t \rangle \cdot s_m \).

**Running Example** We will refer to the following example throughout the remainder of this section.

\[
c = \langle \{i\}, \{a \mapsto m_6, x \mapsto m_1, y \mapsto m_2\} \rangle \\
\cdot \langle \{j\}, \{b \mapsto m_7, x \mapsto m_3, y \mapsto m_4\} \rangle \\
\cdot \langle \emptyset, \{z \mapsto m_5\} \rangle \\
\cdot \text{nil}
\]

where

- \( \text{selfSends}(m_1) = \{a, y\} \)
- \( \text{superSends}(m_1) = \{x\} \)
- \( \text{selfSends}(m_2) = \emptyset \)
- \( \text{superSends}(m_2) = \emptyset \)
- \( \text{selfSends}(m_3) = \{b\} \)
- \( \text{superSends}(m_3) = \emptyset \)
- \( \text{selfSends}(m_4) = \{w\} \)
- \( \text{superSends}(m_4) = \{z\} \)
- \( \text{selfSends}(m_5) = \{a\} \)
- \( \text{superSends}(m_5) = \emptyset \)
- \( \text{selfSends}(m_6) = \emptyset \)
- \( \text{superSends}(m_6) = \emptyset \)
- \( \text{selfSends}(m_7) = \emptyset \)
- \( \text{superSends}(m_7) = \emptyset \)

For conciseness, we will in future omit the trailing \( \cdot \text{nil} \) from all examples.

**Definition 12.** The incremental method dictionary, \( \text{delta}(c) \), of a class \( c \), is:

\[
\text{delta}(c) \overset{\text{def}}{=} \begin{cases} \\
\{\} & \text{if } c = \text{nil} \\
\emptyset & \text{if } c = \langle \phi, d \rangle \cdot c' \\
\end{cases}
\]

**Definition 13.** The superclass, \( \text{super}(c) \), of \( c \) is:

\[
\text{super}(c) \overset{\text{def}}{=} \begin{cases} \\
\text{nil} & \text{if } c = \text{nil} \\
c' & \text{if } c = \langle \phi, d \rangle \cdot c' \\
\end{cases}
\]
Definition 14. The flattening of a class $c$, $\text{flatten}(c)$, is the $\triangleright$ composition of the method dictionaries in its inheritance chain:

$$\text{flatten}(c) \overset{\text{def}}{=} \begin{cases} \{\} & \text{if } c = \text{nil} \\ d \triangleright \text{flatten}(c') & \text{if } c = \langle \phi, d \rangle \cdot c' \end{cases}$$

Definition 15. The method lookup, $c \triangleright x$, of a label $x$ in a class $c$ is:

$$c \triangleright x \overset{\text{def}}{=} \text{flatten}(c)(x)$$

For our running example:

- $\delta(c) = \{a \mapsto m_6, x \mapsto m_1, y \mapsto m_2\}$
- $\text{super}(c) = \langle\{b\}, \{b \mapsto m_7, x \mapsto m_3, y \mapsto m_4\}\rangle \cdot \langle\emptyset, \{z \mapsto m_5\} \rangle$
- $\delta(\text{super}(c)) = \{b \mapsto m_7, x \mapsto m_3, y \mapsto m_4\}$
- $\text{super}(\text{super}(c)) = \langle\emptyset, \{z \mapsto m_5\} \rangle$
- $\delta(\text{super}(\text{super}(c))) = \{z \mapsto m_5\}$
- $c \triangleright y = m_2$
- $\text{super}(c) \triangleright x = m_3$
- $\text{super}(\text{super}(c)) \triangleright x = \bot$

Definition 16. The provided labels, $\text{provided} : C \rightarrow 2^L$, of a class $c$ are defined as:

$$\text{provided}(c) \overset{\text{def}}{=} \{l \in L \mid l \triangleright c \in M\}$$

i.e., the set of all labels that $\text{flatten}(c)$ does not map to $\bot$ or $\top$.

Definition 17. A class $c$ is well-founded if and only if all super-sends in its inheritance chain are bound, i.e., if $\text{superSends}(\delta(c)) \subseteq \text{provided}(\text{super}(c))$ and $\text{super}(c)$ is well-founded. $\text{nil}$ is well-founded by convention.¹

For a particular programming language, a class that is not well-founded may generate run-time errors, or compile-time errors, depending on the philosophy of its designers.

In the running example:

$$\text{provided}(c) = \{a, b, x, y, z\}$$

$c$ is well-founded since the super-send to $x$ in $m_1$ is bound in $\text{super}(c)$.

¹ This definition is slightly more restrictive than necessary; in fact, it does no harm for the superclass to contain a method that has an unbound supersend, provided that this method is not reachable. Of course, such a method is of no practical use.
3.6 Superclass Reachability and Class Equivalence

The \textit{required} labels of a class are more subtle than those of a trait, because we must consider not only self-sends, but also the entire inheritance chain of super-sends. In the running example \textit{c}, it is crucial to capture that the fact that the accessor method \textit{b} is required (and fulfilled) because \textit{m3} is reachable through a chain of super-sends, whereas \textit{w} is not required because \textit{m4} is not reachable.

Thus, to determine which methods are reachable, we see that we must consider not only the methods bound to the provided labels, but methods that may be reached by super-sends from the corresponding methods, methods that may be reached by super-sends from those methods, and so on. We introduce the notation \( c \uparrow \bar{x} \) to mean that it is possible for the method bound to \( \bar{x} \) in class \( c \) to perform a super-send to \( \bar{x} \), and for the method bound to \( \bar{x} \) in this context to perform a supersend .... to \( \bar{x} \). If this occurs, then \( \bar{x} \) will be bound to method \( m \) obtained from \( c \). For example, \( c \uparrow x = \langle m_1, c \rangle \), since the service \( x \mapsto m_1 \) occurs in \( \text{delta}(c) \), but \( c \uparrow b = \langle m_7, \text{super}(c) \rangle \). For convenience, we introduce the notation \( c \uparrow \bar{x} \), which returns just the method reachable by \( \bar{x} \) without its class, so \( c \uparrow x = m_1 \) and \( c \uparrow b = m_7 \).

Reachability for longer chains must take super-sends into account, so \( c \uparrow xx = \langle m_3, \text{super}(c) \rangle \), since \( m_3 \) is reachable by the super-send chain \( xx \). On the other hand, \( c \uparrow xy = \bot \), since \( y \notin \text{superSends}(m) \). Note that although \( m_4 \) is not reachable by any super-send chain starting from \( c \), it is reachable from \( \text{super}(c) \).

\textbf{Definition 18.} \( c \uparrow \bar{x} \), where \( c \in C \) and \( \bar{x} \in \mathcal{L}^+ \), is defined recursively, as follows:

\[
\begin{align*}
\text{nil} \uparrow x & \overset{\text{def}}{=} \langle \bot, \text{nil} \rangle \\
c \uparrow x & \overset{\text{def}}{=} \begin{cases} 
\langle m, c \rangle & \text{if } m = \text{delta}(c)(x) \in \mathcal{M} \\
\text{super}(c) \uparrow x & \text{otherwise}
\end{cases} \\
c \uparrow \bar{xy} & \overset{\text{def}}{=} \begin{cases} 
\text{super}(c') \uparrow y & \text{if } c \uparrow \bar{x} = \langle m, c' \rangle \text{ and } y \in \text{superSends}(m) \\
\langle \bot, \text{nil} \rangle & \text{otherwise}
\end{cases}
\end{align*}
\]

\textbf{Definition 19.} \( c \uparrow \bar{x} \overset{\text{def}}{=} m \), where \( \exists c' \) such that \( c \uparrow \bar{x} = \langle m, c' \rangle \).

\textbf{Theorem 1.} For any class \( c \) and any single message \( x \), \( c \uparrow x = c \gg x \)

\textbf{Proof.} By induction on the length of the super-send chain. The base case where \( c = \text{nil} \) is trivial. The induction step has two cases, where \( x \) is bound in \( \text{delta}(c) \) and where it is not.

\textbf{Definition 20.} A method \( m \in \mathcal{M} \) is reachable from class \( c \), if \( \exists \bar{x} \in \mathcal{L}^+ \) such that \( m = c \uparrow x \).

\textbf{Definition 21.} The set of methods reachable from a class \( c \), is:

\[
\text{reachable}(c) \overset{\text{def}}{=} \{ c \uparrow \bar{x} \mid \bar{x} \in \mathcal{L}^+ \} \setminus \{ \bot \}
\]
In the running example, \( \text{reachable}(c) = \{m_1, m_2, m_3, m_5, m_6, m_7\} \).

**Definition 22.** The set of required labels, \( \text{required}(c) \), of a class \( c \) is:

\[
\text{required}(c) \overset{def}{=} \text{selfSends(reachable}(c)) \setminus \text{provided}(c)
\]

Note that the required labels include the self-sends, not only of the provided methods, but also of all methods reachable through super-sends.

**Definition 23.** A class, \( c \in \mathcal{C} \), is concrete if \( \text{required}(c) = \emptyset \). A class that is not concrete is abstract.

In the example, \( \text{required}(c) = \{a, b, y\} \setminus \{a, b, x, y, z\} = \emptyset \), so \( c \) is concrete. In contrast, \( \text{required}(\text{super}(c)) = \{b, w\} \setminus \{b, x, y, z\} = \{w\} \), so \( \text{super}(c) \) is abstract. Similarly, \( \text{required}(\text{super}(\text{super}(c))) = \{a\} \setminus \{y\} = \{a\} \), so \( \text{super}(\text{super}(c)) \) is also abstract.

Two classes are equivalent if exactly the same methods are reachable by the same super-send chains.

**Definition 24.** A class \( c \) is equivalent to a class \( c' \), \( c \equiv c' \), iff:

\[
\text{provided}(c) = \text{provided}(c') \text{ and } \forall \bar{x} \in \mathcal{L}^+, c \upharpoonright \bar{x} = c' \upharpoonright \bar{x}.
\]

Suppose that:

\[
c' = \langle \{a\}, \{a \mapsto m_6, x \mapsto m_1, y \mapsto m_2, z \mapsto m_3\} \rangle \cdot \langle \{b\}, \{b \mapsto m_7, x \mapsto m_3\} \rangle
\]

Then \( c \equiv c' \) since

\[
\begin{align*}
\text{provided}(c) &= \text{provided}(c') = \{a, b, x, y, z\} \\
\text{c} \upharpoonright x &= c' \upharpoonright x = m_1 \\
\text{c} \upharpoonright y &= c' \upharpoonright y = m_2 \\
\text{c} \upharpoonright a &= c' \upharpoonright a = m_6 \\
\text{c} \upharpoonright b &= c' \upharpoonright a = m_7 \\
\text{c} \upharpoonright z &= c' \upharpoonright z = m_5 \\
\text{c} \upharpoonright xx &= c' \upharpoonright xx = m_3
\end{align*}
\]

and no other methods are reachable.

### 4 Some Examples

In this section, we give the formalizations of some of the examples presented in the traits paper [SDNB02]. For conciseness, we omit the methods to which labels are bound except when there is a method conflict. For example, rather than writing \( \{a \mapsto m_1, b \mapsto m_2, c \mapsto \top\} \) we simply write \( \{a, b, c \mapsto \top\} \).
Fig. 1. (a) shows how a trait TCircle is composed from a trait TGeometry and a nested trait TMagnitude, which is again composed from the trait TEquality. Note that the provided services of the nested traits are propagated to the composite trait (e.g., max, ∼=, and area), and similarly, the unsatisfied requirements of the nested traits (e.g., center and radius) are turned into required methods of the composite trait. In (b), we again use the trait TEquality to specify the comparison behaviour of the trait TColor. (c) shows how a class Circle is specified by composing the traits TCircle, TColor, and TDrawing.
$TEquality = \{\sim =\}$

$TDrawing = \{\text{draw, refresh, refreshOn}\}$

$TGeometry = \{\text{area, bounds, diameter, scaleBy:} \ldots \}$

$TMagnitude = \{<, >, >=, \text{between:and:}, \text{max; min:}\}$ \text{with} $TEquality$

$TColor = \{\text{red, green, blue, hue, saturation, red:} \ldots , =, \text{hash}\}$ \text{with} $TEquality$

$TCircle = \{<, =, \text{hash}\}$ \text{with} $(TMagnitude + TGeometry)$

$Circle = \{(\center, \text{radius, rgb}),$
\{\center, \center, \text{radius, radius; rgb, rgb; initialize, =, hash, drawOn:}\}$
\text{with} $(TColor[\text{colorEquals:} \rightarrow =][\text{colorHash} \rightarrow \text{hash}]$
+ $TCircle[\text{circleEquals:} \rightarrow =][\text{circleHash} \rightarrow \text{hash}]$
+ $TDrawing)$
\text{extends Object}$

\textbf{Fig. 2.} Formal definition of the classes and traits shown in figure 1.

\subsection*{4.1 Graphical objects}

In Section 3 of the traits paper [SDNB02, figure 5], we presented an example that uses traits to represent graphical objects, such as circles and rectangles, in a structured and reusable way. For ease of reference, the figure is reproduced here as figure 1. We now show how the traits and classes given in figure 1 are expressed in our model.

Figure 2 presents formalizations of all the traits and classes that are involved in this example. The first three traits $TEquality$, $TDrawing$, $TGeometry$ are “primitive traits”, which means that they do not make use of any other traits. None of these traits is complete, because they all send messages that do not correspond to provided services. In fact, the required labels of these traits are:

\begin{align*}
\text{required}(TEquality) &= \{=\} \\
\text{required}(TDrawing) &= \{\text{bounds, drawOn:}\} \\
\text{required}(TGeometry) &= \{\text{center, radius, radius}\}
\end{align*}

The trait $TMagnitude$ uses the trait $TEquality$, and implements additional selectors such as $<=$, $<$, $>$, etc.. Note that the trait $TMagnitude$ does not provide the service $=$ required by the trait $TEquality$. This requirement is therefore propagated to the trait $TMagnitude$. Since $TMagnitude$ additionally requires the service $<$, the complete set of requirements is $\text{required}(TMagnitude) = \{<, =\}$.

The trait $TColor$ also uses the trait $TEquality$, but unlike $TMagnitude$, it implements a service $=$, which satisfies the requirement of $TEquality$. In addition,
$TColor$ also implements some color methods such as $red$, $green$, $red$; etc., which require the services $rgb$ and $rgb$.

The trait $TCircle$ is the composition of the traits $TMagnitude$ and $TGeometry$. In addition, it implements the methods $<$, $=; hash$ and thereby satisfies all the requirements for $TMagnitude$. However, it does not satisfy any of the requirements of $TGeometry$ and therefore, $required(TCircle) = required(TGeometry)$.

Finally, we define the class $Circle$, making use of the traits $TColor$, $TCircle$, $TDrawing$. This means that we combine these three traits by using the symmetric composition operator $+$. Since $TColor$ and $TCircle$ provide different methods for $=$ and $hash$, this means that a conflict arises:

$$conflicts(TColor + TCircle + TDrawing) = \{=, hash\}$$

The class $Circle$ resolves this conflict by implementing methods for $=$ and $hash$, which replace the conflicting methods. In order to allow these methods to access the methods from $TColor$ and $TCircle$, we use aliasing to make these methods available in $Circle$ under unambiguous names.

Note that the class $Circle$ satisfies all the requirements of the traits that it uses by defining methods that access its fields $center$, $radius$ and $rgb$.

### 4.2 Read-write synchronization

In Section 5.2 of the traits paper, we describe how we can build a trait that implements simple read-write synchronization in a reusable way. The problem assumes the existence of two classes, $A$ and $B$ that are not related by inheritance and provide unsynchronized read and write methods.

$$A = \langle\{\}, \{read, write, \ldots\}\rangle \cdot A'$$

$$B = \langle\{\}, \{read, write, \ldots\}\rangle \cdot B'$$

Our task is to create synchronized subclasses $SyncA$ and $SyncB$. This can be accomplished by writing a trait $TSyncReadWrite$, that implements a simple synchronization strategy.

$$TSyncReadWrite = \{read, write, acquireLock, releaseLock\}$$

The methods $read$ and $write$ are implemented as shown in Figure 1 of the traits paper:

```plaintext
read
    self acquireLock.
    value := super read.
    self releaseLock.
    ↑ value

write
    self acquireLock.
    value := super write.
    self releaseLock.
```
This means that the method `read` first calls the method `acquireLock`, then it calls the unsynchronized `read` method by using a super-send, and finally it calls the method `releaseLock`. The method `write` is similar. Since both `acquireLock` and `releaseLock` need to access a semaphore, the requirements and super-sends of `TSyncReadWrite` are as follows.

\[
\text{required}(\text{TSyncReadStream}) = \{\text{semaphore}\} \\
\text{superSends}(\text{TSyncReadWrite}) = \{\text{read, write}\}
\]

Now we can use this trait to create subclasses `SyncA` of `A` and `SyncB` of `B`.

\[
\begin{align*}
\text{SyncA} &= \langle\{\text{semaphore}\}, \{\text{semaphore}\} \triangleright \text{TSyncReadWrite}\rangle \cdot A \\
\text{SyncB} &= \langle\{\text{semaphore}\}, \{\text{semaphore}\} \triangleright \text{TSyncReadWrite}\rangle \cdot B
\end{align*}
\]

Since the superclasses `A` and `B` provide methods labeled `read` and `write`, the classes `SyncA` and `SyncB` are well-formed. And because `SyncA` and `SyncB` both provide an accessor method labeled `semaphore` (and the corresponding field), they are also concrete.

## 5 Summary

This paper has introduced a formal model for the mechanism described in the “traits paper” [SDNB02]. It has also formalized the notions of `required` and `provided` methods in an inheritance chain, and has used these notions to explain what is meant by a well-formed class and a concrete class. Notice that these definitions are in fact independent of the trait mechanism, and can be applied to any single-inheritance language.

### References


[SDNB02] Nathanael Schärli, Stéphane Ducasse, Oscar Nierstrasz, and Andrew Black. Traits: Composable units of behavior. Technical Report IAM-02-005, Institut für Informatik, Universität Bern, Switzerland, November 2002. Also available as Technical Report CSE-02-014, OGI School of Science & Engineering, Beaverton, Oregon, USA.